STAT2011 Probability and Estimation Theory – Semester 1, 2021

Sample Quiz 1 - Solutions

1. Suppose that $P(A) = 0.2$ and $P(B) = 0.3$. What does $P(A \cup B)$ equal if A and B are independent?

   $P(A \cup B) = P(A) + P(B) - P(A \cap B) \overset{\text{Independent}}{=} P(A) + P(B) - P(A)P(B) = 0.2 + 0.3 - 0.06 = 0.44$

2. The probability is 0.2 that a tyre will need to be replaced in any given year of driving. What is the probability that a car having four tyres will need to have exactly three tyres replaced in a year?

   $P(X = 3) = \binom{4}{3} 0.2^3 0.8^1 = 0.026$ (3 dec. pl.)

3. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed, if 2 of the men are feuding and refuse to serve on the committee together?

   If we select neither of the 2 men, there are $\binom{5}{3} = 10$ ways of choosing from the men. If we select one of the feuding men, there are $\binom{5}{2} = 10$ ways of choosing from the rest of the men. Thus in total, we have $\binom{5}{3} + 2 \cdot \binom{5}{2} = 30$ (since we can choose either one of the feuding men) of forming the male part of the committee. Combining with the female part, we have $30 \times \binom{5}{2} = 300$ ways of forming the whole committee.

4. The discrete random variable X takes values 0, 1 and 2, and has cumulative distribution function $F_X(x) = (x^2 + 1)/5$ for $x = 0, 1, 2$. What is $P(X = 2)$?

   $P(X = 2) = P(X \leq 2) - P(X \leq 1) = F_X(2) - F_X(1) = 1 - 2/5 = 3/5$

5. For what value of k is the function $f_Y(y) = k(y + 1)$, $-1 \leq y \leq 1$ a probability density function (pdf)?

   $f_Y(y)$ is already non-negative over $[-1, 1]$, thus to be a pdf it remains to have an area under the curve of 1.

   $$\int_{-1}^{1} f_Y(y)dy = \int_{-1}^{1} k(y + 1)dy = k(y^2/2 + y)\Big|_{-1}^{1} = k(1/2 + 1 - 1/2 + 1) = 2k = 1$$

   Thus $k = 1/2$.

6. Country A inadvertently launches six guided rockets – four armed with nuclear warheads – at Country B. In response, Country B fires five antiballistic rockets, each of which will destroy exactly one of the incoming rockets. The antiballistic rockets have no way of detecting, though, which of the six rockets are carrying nuclear warheads. What are the chances that Country B will be hit by exactly one nuclear rocket?

   Here, we can use the hypergeometric distribution to model the problem, where the probability that exactly k nuclear missiles hit is given by

$$\frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}}$$

   with $r = 4$ and $w = 2$, thus $N = 6$. Further, $n = 5$ and $k = 3$, thus

$$\frac{\binom{4}{3}\binom{2}{2}}{\binom{6}{5}} = \frac{4}{6} = 2/3.$$

7. An urn contains three chips numbered 1, 2 and 3. Two are drawn at random without replacement. Let X be the random variable denoting the smaller of the two. What is $E(X)$?

   The draws are equally likely. Since we are only interested in the value of the smaller number in the sample of two, we can list the outcomes as pairs where order does not matter. All the possible outcomes are $(1, 2)$, $(1, 3)$ and $(2, 3)$, thus X is 1, 1 and 2, respectively. It follows $P(X = 1) = 2/3$ and $P(X = 2) = 1/3$. Therefore

$$\mathrm{E}(X) = 2/3 + 2/3 = 4/3.$$

8. Sam takes a ten-question multiple-choice exam where each question has five possible answers. Some of the answers she knows, while others she gets right just by making lucky guesses. Suppose that the conditional probability of her knowing the answer to a randomly selected question given that she got it right is $P(K|R) = 0.8$. How many of the ten questions was she prepared for?

   We determine first $P(R|K) = 1$ and $P(R|K^C) = 1/5$. Then using Baye's theorem, we need to solve the following equation for x,

$$0.8 = P(K|R) = \frac{P(K \cap R)}{P(R)} = \frac{P(R|K)P(K)}{P(R|K)P(K) + P(R|K^C)P(K^C)}$$

$$0.8 = \frac{x/10}{x/10 + (1/5)((10 - x)/10)} = \frac{x}{x + (10 - x)/5}$$

   Thus,

$$\frac{4}{5}x - \frac{4}{25}x + 1.6 = x \Rightarrow x = 4.44 \approx 4$$

9. Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, B be the event that the second toss results in heads, and let C be the event that in both tosses the coin lands on the same side. Which one(s) of the following independence statements are true? Explain.

   (i) A and C are independent. (ii) B and C are independent. (iii) A, B, C are mutually independent.

   Only (i) and (ii) are true.

To see (i), P(A and C) = P({first toss heads} and {both land on the same side}) = P(both land heads) = 1/4. P(A)=1/2, P(C) = P(both land on heads) + P(both land on tails) = 2*(1/2)*(1/2)=1/2. So we have P(A and C) = P(A)P(C).

The argument for (ii) is exactly the same.

For (iii), P(A and B and C) = P(both land on heads) = 1/4. But P(A)=1/2, P(B)=1/2, P(C)=1/2, thus P(A and B and C) does not equal P(A)P(B)P(C). A, B, C are not independent.

10. Give a combinatorial argument for the following identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}, \qquad n \geq k.$$

Consider forming a subset of size k from the numbers 1, 2, ..., n.

The LHS is the total number of different subsets of size k. Another way of counting the subsets is to first fix the largest number in a subset, call it i, and i goes from k (since we need k numbers in each subset) to n. Having selected the largest number i, there are "i-1 choose k-1" ways of choosing the rest of the k-1 numbers. Summing over i, we have the RHS.

Since LHS and RHS are counting the same thing, they must be equal.